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DUAL PARAMETRIZATION OF THE BIFURCATION SET OF A MECHANICAL SYSTEM WITH QUADRATIC INTEGRALS†

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A well-known scheme of topological analysis [1] is extended to the case of three or more integrals which are quadratic functions of the velocity. The bifurcation set is parametrized by Lagrange multipliers, which corresponds to a transition to the dual surface. As an example, the motion of a rigid body in a field with quadratic potential is considered. © 1997 Elsevier Science Ltd. All rights reserved.

1. Let us assume that a dynamical system on the tangent bundle TM of an n-dimensional manifold M with Riemannian metric $\langle\cdot,\cdot\rangle$ has m integrals

$$F_i(\mathbf{v}) \doteq \langle \mathbf{v}, \Gamma_i \mathbf{v}\rangle + W_i(x); \quad \mathbf{v} \in T_x M \tag{1.1}$$

where $\mathbf{v}$ is the velocity, T_xM is the tangent fibre at $x \in M$, Γ_i are the self-adjoint linear fibre operators, and Γ_1 is the identity operator. Consider the integral map $F: TM \to R^m$, $F(\mathbf{v}) = f = (f_1, \ldots, f_m)$, $f_i = F_i(\mathbf{v})$, whose range we denote by Ω, and the map $W: M \to R^m$ defined similarly by the functions $W_i(x)$ and giving the values of F at zero velocity, the set of which we denote by Ω_0.

We adopt the notation λf for the convolution of the vector f with the covector $\lambda = (\lambda_1, \ldots, \lambda_m)$, and the notation λF, $\lambda\Gamma$, λW for the sheaves F_i, Γ_i, W_i with coefficients λ_i.

The object of topological analysis [1] is to describe the integral surfaces $I_f = F^{-1}(f) \subset TM$ and bifurcation set $\Sigma \subset R^m$, which consists of points f corresponding to modifications (surgeries) I_f, i.e. it includes the critical values of the integral map. The critical points of the latter are determined by the dependency condition of the differentials dF_i, which may be represented as $d\lambda F = 0$, where λ_i are the Lagrange multipliers. This condition is invariant [2], that is, the critical points of the sheaf λF form complete motions, which we call steady motions. We will use the quantities λ to parametrize the families of steady motions and the bifurcation surface.

As the partial gradient of the function λF along the fibre T_xM equals $2\lambda\Gamma\mathbf{v}$, the critical points $\mathbf{v}$ of the map F corresponding to a specific λ lie in the kernel of the operator $\lambda\Gamma$. Let $P(\lambda, x) = \det\lambda\Gamma$, $P'(\lambda, x) = \partial P/\partial\lambda_1$, $D(\gamma, \lambda, x) = \det(\lambda\Gamma - \gamma E)$, where $\gamma_i(\lambda, x)$ are the roots of the equation $D = 0$. Since $\lambda\Gamma - \gamma E = (\lambda_1 - \gamma)E + \ldots \lambda_m\Gamma_m$, it follows that $\partial D/\partial\gamma|_{\gamma=0} = -P'(\lambda, x)$. If $P = 0$, one of the roots, say γ_1, must vanish, the kernel of $\lambda\Gamma$ is not zero, and by Viète's theorem $P' = \gamma_2\gamma_3, \ldots \gamma_n$. If $P = 0$ and $P' \neq 0$, the multiplicity of the zero root and the dimensions of the kernel of $\lambda\Gamma$ are greater than one. Then the rank of the matrix $\lambda\Gamma$ is less than $n-1$, that is, all the cofactors P_{ij} of its elements γ_{ij} vanish, so that $dP = \Sigma P_{ij}d\gamma_{ij} = 0$. Thus, when $P' = P = 0$, the differential dP vanishes, and so does the partial differential d_xP.

The functions P and P' define parametrizations by λ of families of functions $P_\lambda(x)$ and $P'_\lambda(x)$ on M. It follows from the foregoing that, given λ, the points x over which the kernel of $\lambda\Gamma$ does not vanish form a surface $Z_\lambda = \{x: P_\lambda(x) = 0\}$, at whose regular points (we shall assume that these are points of general position on Z_λ) $P'_\lambda(x) \neq 0$, so that the kernel of $\lambda\Gamma$ over them is one-dimensional. It can be shown that the surface Z_λ and function λW are invariant with respect to the vectors of the kernel λF. (For natural systems this follows from the equations considered in [3].)

Proposition 1. The vectors $\mathbf{v} \in \ker\lambda\Gamma$ over regular points of the surface Z_λ are critical points of the integral map provided that

$$d\lambda W + \vartheta dP_\lambda = 0; \quad |\mathbf{v}|^2 = \vartheta P'_\lambda \tag{1.2}$$

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Proof. Suppose that λ is fixed, $P_\lambda = 0$ and $dP_\lambda \neq 0$ at the point x_* and the vector $\mathbf{v}_*$ lies in the kernel of $\lambda\Gamma$ over x_*. Then $D = 0$ and $\partial D/\partial\gamma = -P'_\lambda \neq 0$ for $\gamma = 0, x = x_*$. Applying the implicit-function theorem in the neighbourhood if x_*, we obtain a smooth function $\gamma_1(x)$—a simple eigenvalue of the operator $\lambda\Gamma$, $\gamma_1(x_*) = 0$. In that case, in the same neighbourhood a smooth field $\mathbf{v}^*(x)$ of eigenvectors belonging to γ_1, $\mathbf{v}(x_*) = \mathbf{v}_*$, exists. Consider the functions $g(x) = \lambda F(\mathbf{v}(x)) = \gamma_1(x)|\,\mathbf{v}(x)|^2 + \lambda W(x)$ and $G(\mathbf{v}) = \lambda F(\mathbf{v} + \mathbf{v}(x))$, $\mathbf{v} \in TM$. We have $d\lambda F|_{\mathbf{v}=\mathbf{v}(x)} = dG|_{\mathbf{v}=0}$. The partial differentials along T_xM appearing as summands on both sides of this equality vanish at $x = x_*$, since $\lambda\Gamma\mathbf{v}_* = 0$. The remaining term on the right-hand side is the differential of the function $g(x) = G|_{\mathbf{v}=0}$. Therefore, $\mathbf{v}_*$ is a critical vector provided that $dg = 0$ at x_*., which, since $\gamma_1(x_*) = 0$, is equivalent to $|\,\mathbf{v}\gamma_1(x)_*\,|^2 d\gamma_1 + d\lambda W = 0$. On the other hand, $P_\lambda = \gamma_1(x)p(x)$, where $p = \gamma_2\gamma_3 \cdot \ldots \cdot \gamma_n$, and therefore, if $\gamma_1 = 0$, we have $dP_\gamma = p d\gamma_1$ and $P'_\lambda = p$, which implies (1.2) for $-\vartheta = p^{-1}(x_*)|\,\mathbf{v}_*\,|^2$.

The first condition of (1.2) defines the critical points of the restriction $\lambda W|Z_\lambda$ and the values of ϑ at those points. Denote the set of such points at which $\vartheta P'_\lambda \geqslant 0$ by X_λ. If X_λ is connected, then λW takes a constant value on it, which we denote by $\Psi(\lambda)$

$$\Psi(\lambda) = \lambda W(x), \quad x \in X_\lambda \tag{1.3}$$

If X_λ is not connected, Ψ may have a different value on each component.

By Proposition 1, the set X_λ is filled by the trajectories of steady motions at a velocity $\pm\mathbf{v}_\lambda$: $\mathbf{v}_\lambda \in \ker \lambda\Gamma$, $|\,\mathbf{v}_\lambda\,| = \sqrt{\vartheta P'_\lambda}$. By (1.3), the value of the integral λF on these motions is $\Psi(\lambda)$.

The critical points of the map W are defined by the condition $d\lambda W = 0$. By analogy with Proposition 1, the zero vectors over these points are critical points of the map F. Thus, every critical point of the function λW (other than equilibrium points) is the initial point of a steady motion over Z_λ with zero initial velocity; by continuity, this point must lie on Z_λ and the value of λW there equals $\Psi(\lambda)$. The critical values of W form a subset of the critical values of F, and this subset contains $\partial\Omega_0$.

2. Let us assume that the set X_λ, together with the function ϑ are smoothly deformed as λ varies in some domain Λ. Then $\Psi(\lambda)$ is a smooth function in Λ and the vectors $\pm\mathbf{v}_\lambda$ also vary smoothly, running through a subset of critical points of the integral map; denote this subset by S.

Proposition 2. The values $f = F(\mathbf{v}_\lambda)$ of the integral map on S are described by the equation

$$f = \partial\Psi/\partial\lambda \tag{2.1}$$

Proof. By assumption, any vector in S may be embedded in a smooth field $\mathbf{v}(\lambda) = \mathbf{v}_\lambda(x(\lambda))$, $x(\lambda) \in X_\lambda$, $\lambda \in \Lambda$. Let $f = F(\mathbf{v}(\lambda))$, then $\lambda df = d\lambda F = 0$, and, since $\lambda F(\mathbf{v}(\lambda)) = \lambda W(x(\lambda))$, it follows from (1.3) that $\lambda f = \Psi$, whence, after differentiation, since $\lambda df = 0$, we obtain (2.1).

Since $\Psi(\lambda)$ is a homogeneous function, it follows from (2.1) that $\lambda f = \Psi$ and $\lambda df = 0$, that is, λ is the normal and $\Pi_\lambda = \{f\colon \lambda f = \Psi\}$ is the tangent plane to $F(S)$ at the point (2.1). The tangent set $\{\Pi_\gamma, \lambda \in \Lambda\}$ is identified with the surface $\{\lambda_1 : \ldots : \lambda_m : -\Psi(\lambda)\}$ in the adjoint projective space, which is called the space dual to $F(S)$ [4]; accordingly, the parametrization of the section $F(S) \subset \Sigma$ by formula (2.1) is called a dual parametrization and the domain Λ is called a dual bifurcation domain (DBD).

For all λ in a given DBD, the signature of the matrix $\lambda\Gamma$ over the regular points of the connected part of X_λ is the same, since $\gamma_2 \ldots \gamma_n = P' \neq 0$. Those DBDs for which $\lambda\Gamma$ is positive-semidefinite or negative-semidefinite over $x \in X_\lambda$ will be called domains of definiteness and denoted by Λ^+ or Λ^-, respectively.

We will limit ourselves to the case in which Σ is exhausted by the critical values of the integrals (for example, if M is a compact set) and all its smooth parts have the form $F(S)$ for sets S as described above. Then the map (2.1), denoted henceforth by $f(\lambda)$, defines a dual parametrization of the bifurcation surface.

Remark. For every point f, the normals to the planes Π_λ containing it form a surface $\Theta_f = \{\lambda\colon \lambda f = \Psi(\lambda)\}$ in certain DBDs. The envelope K_f of these planes is a cone with apex f and generators parallel to $\mathrm{grad}(\lambda f - \Psi(\lambda)) = f - f(\lambda)$, $\lambda \in \Theta_f$, that is, normal to Θ_f, so that K_f and Θ_f are mutually dual cones. Expressing (2.1) as $\mathrm{grad}(\lambda f - \Psi(\lambda)) = 0$, we observe that the modifications Θ_f occur at $f \in \Sigma$, that is, in the cases considered here they correspond to bifurcations I_f.

3. Let us consider the map $\rho\colon I_f \to M$ defined as the composition of the embedding $I_f \to TM$ and the bundle projection $TM \to M$. The image $\rho(I_f) = M_f$ is the domain of possible motion (DPM) for the given values of the integrals. The set of critical values of ρ is known as the generalized boundary of the DPM [5]; it includes the boundary ∂M_f of the DPM and is denoted by δM_f. Over points $x \in \delta M_f$ one has modifications of the section $\rho^{-1}(x) = I_f \cap T_xM$—sets of possible velocities.

Let Ω_x and $\delta\Omega_x$ be the image of a set of critical values of the restriction $F \mid T_xM$. If $\mathbf{v}$ is a critical point of the map ρ, then necessarily $\lambda\Gamma\mathbf{v} = 0$, i.e. it is also critical for $F \mid T_xM$. Thus, the conditions $x \in M_f$ and $x \in \delta M_f$ are equivalent to the conditions $f \in \Omega_x$ and $f \in \delta\Omega_x$. The map $F \mid T_xM$ is homogeneous up to the term $W(x)$, and therefore the sets Ω_x and $\delta\Omega_x$ are conical with apex $f = W(x)$.

The cone $\delta\Omega_x$, which contains the boundary $\partial\Omega_x$, is described by a dual parametrization with correction for homogeneity. The values of λ form a cone $X_x = \{\lambda : P(\lambda, x) = 0\}$, the role of Λ is played by its regular domains, defined by the condition $P'(\lambda, x) \neq 0$.

Proposition 3. The values of $f \in \delta\Omega_x$ corresponding to regular points $\lambda \in X_x$ are described by the conditions

$$f - W(x) = \mu\partial P/\partial\lambda, \quad \mu\partial P/\partial\lambda_1 \geqslant 0, \quad P = 0 \tag{3.1}$$

with an undetermined factor μ.

The proof is analogous to that of Proposition 2, allowing for the fact that the function $\Psi(\lambda) = \lambda W(x)$ is linear and the differential $d\lambda$ must satisfy the condition $dP = 0$. The inequality follows from the fact that Γ_1 is positive.

Let Λ_x^+ and Λ_x^- be the closures of the domains of λ values for which the operator $\lambda\Gamma$ is positive-definite and negative-definite, respectively, over x. Denote their boundaries, which are contained in X_x, by X_x^+ and X_x^-, respectively; in both of them, $\lambda\Gamma$ is semidefinite. Clearly, the pairs $\Lambda_x^\pm$ and $X_x^\pm$ are symmetrical about zero.

Proceeding as at the end of Section 2, we conclude that the planes $\Pi_{\lambda x} = \{f: \lambda f = \lambda W(x)\}$ for $\lambda \in X_s$ are tangent to $\delta\Omega_x$. If $\lambda \in \Lambda^\pm$, they are support planes for Ω_x, and if $\lambda \in X_x^\pm$, they are tangent to $\partial\Omega_x$. Indeed, since $\lambda F(\mathbf{v}) - \lambda(W(x) = \langle \mathbf{v}, \lambda\Gamma\mathbf{v}\rangle$, it follows that $\pm\lambda\Gamma \geqslant 0$ if and only if $\pm(\lambda f - \lambda W(x)) \geqslant 0$ for all $f = F(\mathbf{v})$, $\mathbf{v} \in T_xM$, with equality for $f \in \partial\Omega_x$, $\lambda \in X_x^\pm$, $\lambda\Gamma\mathbf{v} = 0$. Thus, the cone $\Lambda_x^\pm$ is dual to Ω_x [6], and hence $\Lambda_x^\pm$ is convex.

In what follows we will consider the case in which all the sets Ω_x are convex. This is the case, for example, when $m = 3$.

To prove this, one shows that the section of Ω_x by a plane $\{f_1 = \text{const}\}$ is convex. Any straight line in the section may be reduced, by a linear substitution of f_2 and f_3, to the form $\{f_1 = \text{const}, f_2 = \text{const}\}$. Its pre-image in T_xM—the intersection of two quadratic surfaces—is the union of a pair of connected components symmetrical about zero. Hence the set of values $f_3 = F_3(\pm\mathbf{v})$ on the pre-image is a single interval, as required.

Proposition 4. The domains of possible motion have the form

$$M_f = \{x: \forall\lambda \in X_x^+,\ \pm(\lambda f - \lambda W(x)) \geqslant 0\} \tag{3.2}$$

Proof. By convexity, Ω_x is dual to Λ_x^+ [6], and therefore the condition that $\lambda \in X_x^\pm = \partial\Lambda_x^\pm$ should imply $\pm(\lambda f - \lambda W(x)) \geqslant 0$ is necessary and sufficient for $f \in \Omega_x$, that is, for $x \in M_f$.

The boundary of Ω is obviously a subset of Σ.

Proposition 5. The smooth parts of $\partial\Omega$ are the images of the domains of definiteness under a dual parametrization, in such a way that the normals λ in Λ^+ are inward and those in Λ^- outward with respect to Ω.

Proof. Let $f(\lambda) \in \partial\Omega$. Since $\Omega = \cup\Omega_x$, it follows that $\partial\Omega$ is tangent to $\partial\Omega_x$ at the point $f(\lambda)$ for some $x \in X_\lambda$. Then λ is normal to $\partial\Omega_x$, and the tangent plane $\Pi_{\lambda x}$ is a support plane by convexity; hence $\lambda \in X^\pm{}_x$ and so $\lambda \in \Lambda^\pm$. It follows from (2.1) and (1.3) that $\lambda f(\lambda) = \lambda W(x)$, and therefore, if $f = f(\lambda) + df$, then $\lambda df = \lambda f - \lambda W(x)$, and if $\pm\lambda df > 0$, the point f lies on the same side of $\Pi_{\lambda x}$ as Ω_x, that is, the normal $\pm\lambda$ is inward for Ω_x and Ω.

4. Let us consider the motion of a rigid body with a fixed point in a field with a quadratic potential [7]. Let $Q = (\alpha_{ij})$ be the matrix of the transformation from principal axes of inertia to principal axes of potential, and let $I = \text{diag}(I_i)$ and $A = \text{diag}(a_i)$ be the principal matrices of inertia and potential. Assume that $I_1 > I_2 > I_3$, $a_1 > a_2 > a_3 > 0$. Let $J_i = I_{i+1}I_{i-1}$, $b_i = a_{i+1}a_{i-1}$, $i(\text{mod}3)$. Define matrices

$U = (u_{ij}) = QAQ^{-1}, J = \mathrm{diag}(J_i), B = \mathrm{diag}(b_i), V = (v_{ij}) = QBQ^{-1}$. Three integrals—linear combinations of those described in [7]—have the form (1.1), where $\mathbf{v} = \omega$ is the angular velocity, the metric is defined by the inertia tensor, and the matrices of operators in its principal axes and the functions $W_i(Q)$ are as follows:

$$\Gamma_2 = I, \quad \Gamma_3 = UI; \; W_1 = \mathrm{tr}UI, \; W_2 = -\mathrm{tr}UJ, \; W_3 = \mathrm{tr}VJ$$

Let us number the permutations σ of the numbers 1, 2, 3 as follows: σ_0 is the ideal permutation, $\sigma_1 = (123)$ is a cyclic permutation, $\sigma_2 = \sigma_1^{-1}$, $\sigma_{3,4,5}$ are transpositions, and $(ij) = \sigma_{i+j}$. Define vectors in the space R^3 by $c_{ij} = (1, I_i, I_i, a_j)$, $i, j = 1, 2, 3$ and points c_k, $k = 0, \ldots, 5$ by

$$c_k = \sum_{i=1}^{3} (I_i a_{\sigma(i)}, -J_i a_{\sigma(i)}, J_i b_{\sigma(i)}), \quad \sigma = \sigma_k$$

As local coordinates of a point Q we take the values $w_i = W_i(Q)$. Let $w = (w_1, w_2, w_3)$. Then $\{w\} = \Omega_0$ and

$$P(\lambda, w) = \lambda_1\lambda_3(\lambda w - L(\lambda)), \quad L = \lambda c_k - (\lambda_1\lambda_3)^{-1} \sum_{i=1}^{3} \lambda c_{i\sigma(i)}$$

for any $\sigma = \sigma_k$ ($k = 0, \ldots, 5$). Since Z_λ is a level surface $L(\lambda)$ of the function λW, it follows by (1.3) that $\Psi(\lambda) = L(\lambda)$.

We carry over the notation Θ_f or to the projective curve of third order $\{\lambda : \lambda f = L(\lambda)\}$.

Then X_w coincides with Θ_f when $f = w$. The set of points f such that the curve Θ_f has a singularity is defined by the property that the discriminant of the curve vanishes, that is, the set is an algebraic surface of order 12 [8], which we denote by Δ. The set Σ lies on Δ (see the remark at the end of Section 2).

To describe Z_λ we will find the critical points of the sheaves λW. Omitting the calculations, we present the results for the case $a^{-1} < r < a$, $a = (a_1 - a_2):(a_2 - a_3)$, $r = (J_2 - J_1):(J_3 - J_2)$. It is sufficient to take the values of λ in the plane $\{\lambda_3 = 1\}$, which is shown in Fig. 1 divided into domains by the straight lines $l_{ij} = (\lambda : \lambda c_{ij} = 0\}$. The points of intersection of the straight lines $l_{i\sigma(i)}$ for $\sigma = \sigma_k$ are indicated by the digits $k = 0, \ldots, 5$. For λ in the domains $\Lambda_0^{\pm}$, Λ_1, Λ_2, 16 critical points of λW exist, defined by the equations

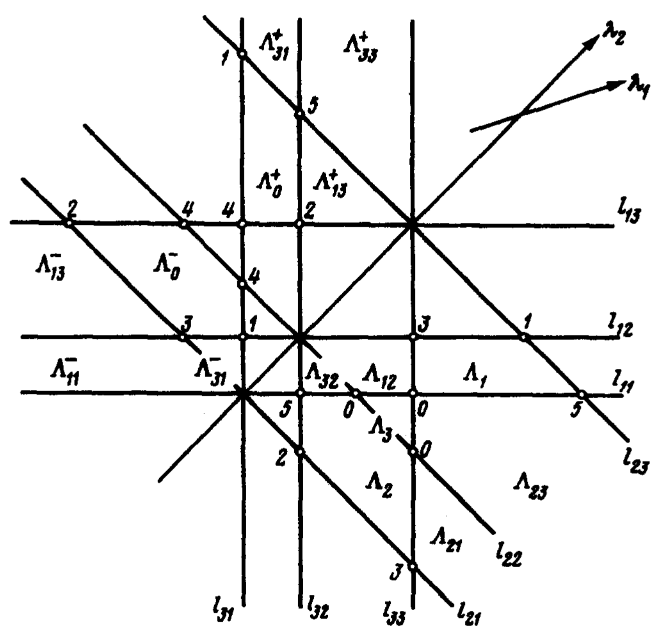

Fig. 1.

$$\alpha_{ij}^2(\lambda)=\frac{\lambda c_{i(j+1)}\lambda c_{i(j-1)}\lambda c_{(i+1)j}\lambda c_{(i-1)j}}{(I_i-I_{i+1})(I_i-I_{i-1})(a_j-a_{j+1})(a_j-a_{j-1})(\lambda_1\lambda_3)^2}$$

i, j (mod 3), and differing in the signs of α_{ij}, we give them a common notation Q_λ. The other critical points for $\lambda \in I_{ij}$ fill out curves $C_{ij} = \{Q : \alpha_{ij} = \pm 1\}$. Among them are 24 equilibrium points, where $\alpha_{i\sigma(i)} = \pm 1, w = c_k$ for $\sigma = \sigma_k, dW_i = 0$ ($i = 1, 2, 3; k = 0, \ldots, 5$).

The form of Z_λ will be deducted from the modifications of the level sets λW as the level values pass through the critical values [9]. Since the first condition (1.2) is identically valid for $\vartheta = (\lambda_1\lambda_3)^{-1}$, the set X_λ is the part of Z_λ on which sign $P'_\lambda = -$sign $\lambda_1\lambda_3$. On Z_λ the sign of $P'_\lambda = \gamma_2\gamma_3$ (for $\gamma_1 = 0$) is found from the signature of $\lambda\Gamma$ in the domains separated by Z_λ. We deduce that for λ in the unlabelled domains in Fig. 1 either Z_λ or X_λ is empty. For $\lambda \in \Lambda_0^\pm, \Lambda_1, \Lambda_2$, the set X_λ is formed by four of the eight components of Z_λ which are diffeomorphic to spheres punctured at four points and glued to Q_λ at the punctures. For the other labelled domains, the sets X_λ are identical with Z_λ, are diffeomorphic to a pair of tori and, as λ passes through I_{ij}, are modified according to C_{ij}. Thus, the labelled domains in Fig. 1 are DBDs; the meaning of the superscript plus or minus was indicated previously. The pieces of the boundary $\Lambda_{\cdots}^{\cdots}$ are denoted by $\partial_{ij}\Lambda_{\cdots}^{\cdots} = l_{ij} \cap \partial\Lambda_{\cdots}^{\cdots}$.

The coordinates ω_i of the velocity of a steady motion along X_λ satisfy the conditions

$$\omega_1 : \omega_2 : \omega_3 = u_1^{-1} : u_2^{-1} : u_3^{-1} \tag{4.1}$$
$$u_i = (\lambda_1 + I_i\lambda_2)u_{jk} - I_i\lambda_3 v_{jk}, \quad i \neq j \neq k \neq i$$

(In the equation $\lambda\Gamma\omega = 0$, multiply the ith row by $u_{jk}, i \neq j \neq k \neq i$, single out the term diag(u_i) in the new matrix and note that $u_i\omega_i$ does not depend on i.) The three u_is vanish simultaneously only at points Q_λ where $\omega = 0$, or when $\lambda \in l_{ij}$ on the curves C_{ij} along which the steady motions occur with arbitrary initial ω_i and with $\omega_j = 0$ ($j \neq i$); these motions correspond to the values $\lambda_1 = 0$ or $\lambda_3 = 0$.

The values of the integrals on motions along C_{ij} form nine rays T_{ij} parallel to c_{ij}, each containing two points $c_k, j = \sigma_k(i)$, and ending in one of them. These rays (parts of the self-intersection edges of the surface Δ) are shown schematically in Fig. 2; they form the one-dimensional skeleton of the set Σ. The two-dimensional components of Σ are diffeomorphic images of the domains $\Lambda_{\cdots}^{\cdots}$ in Fig. 1 under the map $f(\lambda)$; denote the image $f(\Lambda_{\cdots}^{\cdots})$ by $\Sigma_{\cdots}^{\cdots}$ with the same indices. These images are glued to the skeleton by continuation of $f(\lambda)$ to the boundary of the DBD. When that is done the vertices labelled k go into c_k, points with $\lambda_1 = 0, \infty$ go to infinity, $\partial_{ij}\Lambda_{\cdots}^{\cdots}$ ($i \neq j$) are mapped monotonically into T_{ij} and $\partial\Lambda_{\cdots}^{\cdots}$ doubly into T_{ii}: $\partial_{ii}\Lambda^\pm{}_{ii}$ onto $[e_i, \infty)$, $\partial_{22}\Lambda_0^-$ onto $[c_4, e_2]$ and $\partial_{ii}\Lambda_3$ onto $[c_0, d_i]$. These segments of the rays T_{ii} will be called inner edges of the cells $\Sigma\Lambda_{\cdots}^{\cdots}$ glued to them (note that there are no two-dimensional cells glued to the segments $(d_i, e_i) \subset T_{ii}$). Thus, the totality of DBDs in Fig. 1 is an "unfolded" image of the bifurcation surface, which demonstrates the relative positions of its parts.

The partition of the domain Ω by the surface Σ consists, besides Ω_0, of four infinite domains Ω_{ij} ($i, j = 1, 3$), one of which contains the finite domain Ω_1. To avoid graphical difficulties, we will present separate schematic depictions of the parts of Σ bounding these domains (for the relationship chosen among I_i, a_i).

Figure 2 shows the boundary of Ω_0; the "invisible" faces Σ_1 and Σ_0 are not shown. Figures 3 and 4 show the boundaries of $\Omega_{ij}, i \neq j$ and Ω_{ii}; here $k = i + 1$(mod 3), the upper index of the plus and minus

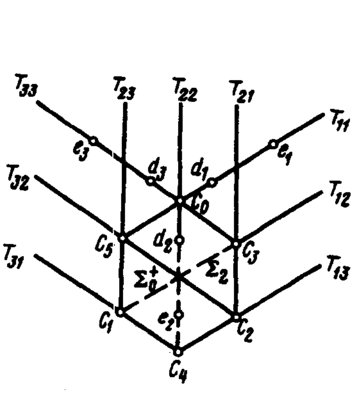

Fig. 2.

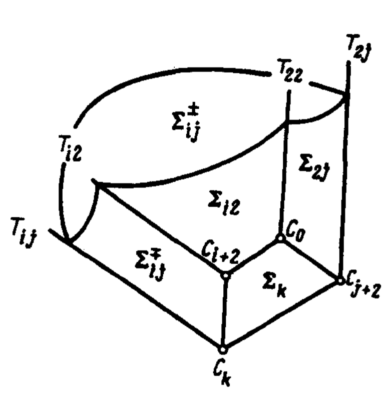

Fig. 3.

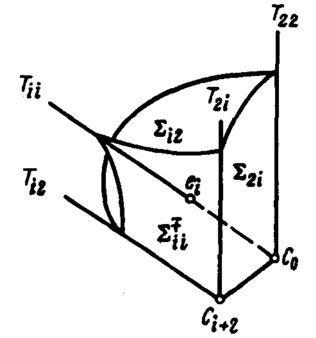

Fig. 4.

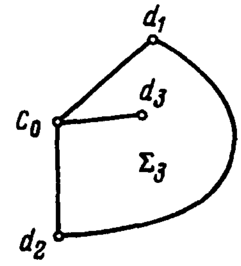

Fig. 5.

superscripts in the symbol $\Sigma^{\cdots}_{\cdots}$ is taken when $i = 1$, the inner edges $[c_0, d_i]$ of the faces Σ_{12}, Σ_{23} and Σ_2 are not shown. Figure 5 represents the closed surface Σ_3 with inner edges $[c_0, d_i]$ it bounds the domain Ω_1, which is contained in Ω_{13}. By Proposition 5, the pieces Σ_0^+, Σ_{ij}^+ and Σ_0^-, Σ_{ij}^- form the upper and lower parts of $\partial\Omega$, respectively, relative to the f_1 axis.

Since $\partial L \partial\lambda = f(\lambda)$, condition (3.1) becomes

$$f - w = \tau(f(\lambda) - w), \quad \lambda w = L(\lambda) \tag{4.2}$$

with an indefinite multiplier τ, which means that the points f, w, $f(\lambda)$ are collinear. Multiplying the first equality of (4.2) by λ we see, by virtue of the second equality and the identity $\lambda f(\lambda) = L$ (which follows from the homogeneity of $L(\lambda)$), that $\lambda(f - w) = 0$, that is, the second equality in (4.2) can be replaced by $\lambda f = L$. Hence it follows that the cone $\delta\Omega_w$ is half $(f_i \geqslant w_i)$ of the cone K_w dual to the set X_w, and the generalized boundary δM_f is the intersection with Ω_0 of the half $(w_i \leqslant f_i)$ of the cone K_f dual to Θ_f (cf. the remark at the end of Section 2).

Consider the domains Λ_f^+, Λ_f^-, $\Lambda_f^\pm = \{\lambda : \pm(\lambda f - L) \geqslant 0\}$ in the plane $\{\lambda_3 = 1\}$; they are bounded by the curve Θ_f and the λ_2 axis. Since $\lambda w = L$ on X_w, we can rewrite (3.2) as

$$M_f = \{w : X_w^\pm \subset \Lambda_f^\pm\} \tag{4.3}$$

Note that condition (4.2), which expresses the membership relation $w \in \delta M_f$, means that X_w is tangent to Θ_f, while the membership relation $w \in \partial M_f$ means, by (4.3), that $X_w^\mp$ is tangent to $\partial\Lambda_f^\mp$.

Since the problem is integrable, the description of the integral surfaces reduces to indicating the number of component tori in each. To that end, we establish the form of M_f and the nature of the projection $I_f \to M_f$ for near-critical values of f.

All the curves Θ_f, have three common points at $\lambda_1 = 0$, $\lambda_2 = -a_i$ and three at infinity on the directions l_{ij}. The form of Θ_f may be established at $f = c_k$ (a triple of straight lines $l_{i\sigma(i)}$, $\sigma = \sigma_k$), at $f \in T_{ij}$ (a straight line l_{ij} and a hyperbola); by the Remark in Section 2, their form for other fs follows by continuity. Let $f_* = f(\lambda_*)$, $\lambda_* \in \Lambda_{11}^-$. Denote the corresponding sets by M_*, Θ_*, Z_*. Let file in Ω_{11} near f_*. Figure 6 shows

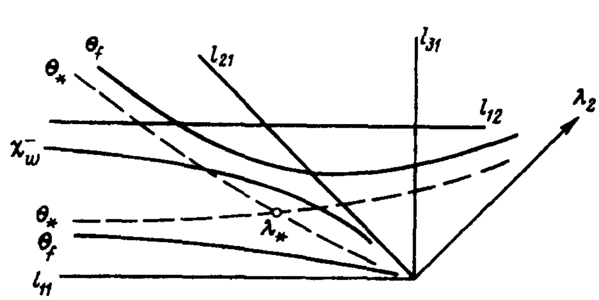

Fig. 6.

segments of the curves X_w^-, Θ_j and Θ_* (dashed) near λ_*—the nodes of Θ_*, and the domains Λ^-_f, Λ^{-*} lie between the upper and lower branches of Θ_f, Θ_*.

It is clear that a necessary condition for $X_w^- \subset \Lambda^-_*$ is that $\lambda_* \in X^-_w$, that is, $w \in Z_*$, and a necessary condition for $X_w^- \subset \Lambda_f^-$ is that the curve X_w pass near λ_*, so that M_f lies near Z_*. Since $Z_* \subset M_*$, it follows that $M_* = Z_*$ and, by (4.3), the relation $\lambda_* \in X_w^-$ is a sufficient condition for $X^{\pm}{}_w \subset \Lambda^{\pm}{}_f$. By continuity, taking into consideration the possible nature of the intersection of the algebraic curves X_w and Θ_f with six common points, one can show that a sufficient condition for $X^{\pm}{}_w \subset \Lambda^{\pm}{}_f$ is that the branch $X_w{}^-$ should pass between the branches Θ_f in Fig. 6; their points of tangency at $w \in \delta M_f$ form sets Θ'_f and Θ''_f on the upper and lower branches, respectively, of Θ_f near λ_*. The other branches of the curve X_w do not pass through Λ_{11}^-; therefore, first, $\partial M_f = \partial M_f$, and second, if $w \in M_f$, the quantity $\lambda w - L(\lambda)$ is non-positive on Θ'_f and non-negative on Θ''_f. Consequently, M_f lies at the intersection of the following domains Ω'_f and Ω''_f:$\Omega'_f = \cap\Omega^-{}_\lambda$, $\lambda \in \Theta'_f$, $\Theta''_f = \cap\Omega^+_\lambda$, $\lambda \in \Theta''_f$; $\Omega^{\pm}_\lambda = \{w{:}\pm(\lambda w - L) \geqslant 0\}$. Since the boundaries of these domains are defined by the same condition of the tangency of X_w with Θ_f as ∂M_f, it follows that $M_f = \Omega'_f \cap \Omega''_f$ and $\partial M_f = \partial\Omega'_f \cup \Omega''_f$.

Shifting each point of Z_* along a vector field transverse to Z_* to a point with coordinates w_i such that X_w is tangent to Θ_f, we obtain a continuous deformation of Z_* into $\partial\Omega'_f$ and $\partial\Omega''_f$. Thus, M_f has two components, each bounded by a pair of tori and contractible to a component of the surface Z_*. Each boundary point of M_f is the image under projection from I_f of two antipodal vectors of the form (4.1), and each interior point is similarly the image of two antipodal pairs of nearby vectors. Hence I_f consists of four three-dimensional tori, each pair of which projects into two components of a DPM.

This is the form of I_f for Ω_{33}, Ω_{13}, Ω_{31}. Proceeding in analogous fashion for Ω_1, one can deduce that the integral surfaces consist of eight tori: four that project onto a DPM, as described above, and two more pairs that project onto subsets of components of a DPM of the same form. For Ω_0, the four tori comprising I_f project, one each, into the four components of the DPM bounding each pair of surfaces homeomorphic to a sphere.

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